Pergamon

J. Appl. Maths Mechs, Vol. 60, No. 2, pp. 205–216, 1996

0021–8928/96 $24.00+0.00

S0021–8928(96)00027-5

THE STABILITY OF HEREDITARY SYSTEMS OF NEUTRAL TYPE†

V. B. KOLMANOVSKII

Moscow

(*Received* 30 *April* 1995)

A method of investigating the stability of systems of neutral type based on the use of Lyapunov's second method is proposed. To illustrate the method, stability conditions are established for a predator–prey system and a power transmission bypass-line.

1. STATEMENT OF THE PROBLEM

The term "hereditary systems of neutral type" refers to systems whose rate of evolution may depend on previous states and rates of the system [1–3]. Such systems are used to simulate a variety of real processes, such as the unsteady motion of bodies in a continuous medium (the phenomenon of aero-auto-elasticity) [4], the control of turbines when there are hydro-shock effects [5], oscillations in long transmission lines [6], the interaction of populations [7, 8], and many others [3, 9].

One of the major problems in the field, commonly encountered in applications, arises in the study of stability and is as follows. Let $\mathbb{R}^n$ be an n-dimensional real linear space equipped with some norm $|\cdot|$. Consider the following equation of neutral type

$$\dot{x}(t) = F(t, x_t, \dot{x}_t), \quad t \geqslant t_0 \tag{1.1}$$

$$x_{t_0} = \psi, \quad \dot{x}_{t_0} = \dot{\psi} \tag{1.2}$$

with $x(t) \in \mathbb{R}^n$ and $x_t = x(t+\theta)$, where for any fixed t the argument θ can take all values from $-\infty$ to 0, and the initial function ψ: $(-\infty, 0] \to \mathbb{R}^n$ is absolutely continuous. It is assumed that the functional $(t, \psi, \varphi) \to F(t, \psi, \varphi)$ is defined in $[t_0, \infty) \times C(-\infty, 0] \times L_\infty(-\infty, 0]$, continuous in $[t_0, \infty) \times C(-\infty, 0] \times L^N_1(-\infty, 0]$ for all $N > 0$ and, for any bounded set $K \subset C(-\infty, 0] \times L_\infty(-\infty, 0]$, constants $\varepsilon > 0$ and $L > 0$ exist such that

$$|F(t,\psi_1,\varphi_1) - F(t,\psi_2,\varphi_2)| \leqslant L[\sup_{\theta\leqslant 0}|\psi_1(\theta) - \psi_2(\theta)| +$$
$$+ \operatorname{vrai\,sup}_{-\infty<\tau\leqslant-\varepsilon}|\varphi_1(\tau) - \varphi_2(\tau)|] + l \operatorname{vrai\,sup}_{-\varepsilon\leqslant t\leqslant 0}|\varphi_1(t) - \varphi_2(t)|$$

where $t \geqslant t_0$, $l \in [0, 1)$, $(\psi_i, \varphi_i) \in K$ $(i = 1, 2)$, and $L^N_1(-\infty, 0]$ is the subspace of functions in $L_\infty(-\infty, 0]$ with finite $L_1(-\infty, 0]$-norm whose L_∞-norm is at most N. Under these assumptions one can establish (see [9]) a (local) existence and uniqueness theorem for the solution of problem (1.1), (1.2), where a solution is defined as an absolutely continuous function of x satisfying (1.1), (1.2) almost everywhere. As usual in stability theory, we assume without lose of generality that

$$F(t, 0, 0) \equiv 0, \quad t \geqslant t_0 \tag{1.3}$$

Definition. The trivial solution of problem (1.1)–(1.3) is said to be

1. stable if, for any $\varepsilon > 0$, a $\delta(\varepsilon) > 0$ exists such that $|x(t)| < \varepsilon$, $t \geqslant t_0$ for any initial conditions that satisfy the inequality $\sup_{\theta\leqslant 0} |\psi(\theta)| + \operatorname{vrai\,sup}_{\theta\leqslant 0} |\dot{\psi}(\theta)| < \delta(\varepsilon)$;
2. asymptotically stable if it is stable and $\lim x(t) = 0$, $t \to \infty$ for any initial conditions in some domain in $C(-\infty, 0] \times L_\infty(-\infty, 0]$.

†*Prikl. Mat. Mekh.* Vol. 60, No. 2, pp. 210–222, 1996.

One way to study the stability of the trivial solution of system (1.1) is Lyapunov's second method. For delay systems, in which $F(t, x_t, \dot{x}_t) \equiv F(t, x_t)$, general stability theorems were formulated [10] in terms of the existence of positive definite Lyapunov functionals which depend on x_t; these theorems were subsequently modified in various directions [2, 3, 7–9, 11]. They can be extended without substantial changes to system (1.1). This formal extension, however, is of limited applicability, because the right-hand sides of the system also depend on previous values of the velocities $\dot{x}_t$. As a result, the general theorems of Lyapunov's second method are formulated for systems of neutral type (1.1) in terms of the existence of either degenerate functionals or functionals that depend on both x_t and $\dot{x}_t$ (see, for example, [3, 9]).

Functionals of these kinds have actually been constructed for some specific systems, and stability conditions have been established with their help that depend directly on the characteristics of the equations themselves [2, 3, 7–9]. In addition, the form of the functionals enables one to detect a certain link between these functionals and Lyapunov functions for suitably chosen ordinary equations.

In what follows we propose a procedure for constructing Lyapunov functionals V in the form $V = V_1 + V_2$ for equations of neutral type (1.1); this procedure was considered in [12] for the special case of delay equation (1.1) (i.e. $F(t, x_t, \dot{x}_t) \equiv F(t, x_t)$).

The procedure is as follows.

1. Transform the right-hand side F of Eq. (1.1) so that it is expressed as the sum of two terms, of which the first depends only on the present state of the system, i.e.

$$\dot{x}(t) = F(t, x(t), \dot{x}(t)), \quad F = F_1(t, x(t)) + F_2(t, x_t, \dot{x}_t) \tag{1.4}$$
$$F_1(t, 0) = 0, \quad F_2(t, 0, 0) = 0$$

2. Delete F_2 from the transformed equation to obtain an auxiliary ordinary equation $\dot{y}(t) = F_1(t, y(t))$, and construct a Lyapunov function $v(t, y)$ for that equation.

3.Replace the second argument of $v(t, y)$ by a certain function, depending on the transformation used in the first step. Specifically: if

$$F_2(t, x_t, \dot{x}_t) = F_3(t, x_t, \dot{x}_t) + \frac{d}{dt} F_4(t, x_t) \tag{1.5}$$

then $V_1(t, x_t) = v(t, z(t))$, where $z = x(t) - F_4(t, x_t)$. But if the component F_4 in representation (1.5) vanishes, then $V_1(t, x_t) = v(t, x(t))$. Now, adding to V_1 the component $V_2 = V_2(t, x_t, \dot{x}_t)$ in such a way as to satisfy the requirements of some stability theorem for system (1.1), we obtain a functional $V = V_1 + V_2$. The use of the functional $v(t, z(t))$ may necessitate a stability analysis for the trivial solution of the functional equation $z(t) = 0$.

We stress that the individual steps of the procedure may be implemented in more than one way—a feature that should be used to expand the stability domains obtained. Moreover, the procedure not only provides a unified formal approach to the construction of a series of already known functionals for equations of neutral type, but also enables one to investigate the stability of specific systems.

The individual steps of the procedure will be clarified by an example. Let $\dot{V}$ denote the right upper derivative of the functional V along trajectories of system (1.1).

Example 1. Consider the scalar equation

$$\dot{x}(t) = -a\, x(t) + b\, \dot{x}(t-h), \quad t \geqslant 0 \tag{1.6}$$

where $a, b, h \geqslant 0$ are given constants, $|b| < 1$.

The right-hand side of Eq. (1.6) is already expressed in the form of (1.3) when $F_1 = -ax(t)$, $F_2 = b\dot{x}(t-h)$. We may therefore proceed to step 2. The auxiliary ordinary equation is $\dot{y}(t) = -ay(t)$, for which the Lyapunov function v may be taken as equal to $v = y^2$. The realization of step 3 depends on the representation (1.5).

Suppose that in (1.5) $F_3 = 0$ and $F_4 = bx(t-h)$. Then $V_1 = (x(t) - bx(t-h))^2$. To construct the component V_2 of the functional $V = V_1 + V_2$, we calculate $\dot{V}_1$

$$\dot{V}_1 = -2ax(t)(x(t) - bx(t-h)) = -2ax^2(t) + 2abx(t)x(t-h) \leqslant (-2a + |ab|)x^2(t) + |ab|x^2(t-h)$$

Thus, if one puts

$$V_2 = |ab| \int_{t-h}^{t} x^2(\tau)\, d\tau$$

then $\dot V \leqslant 2(-a + |ab|)x^2(t)$. Hence $\dot V$ is negative definite if

$$a > 0, \quad |b| < 1 \tag{1.7}$$

Since the functional V as constructed is only non-negative, we shall require in addition that the zero solution of the difference equation $x(t) - bx(t-h) = 0$ must be asymptotically stable. A necessary and sufficient condition for this is that $|b| < 1$. Thus, the functional V yields (1.7) as asymptotic stability conditions for system (1.6).

We will now consider another implementation of the procedure. Suppose that the auxiliary equation is $\dot y(t) = -ay(t)$, the Lyapunov function for which may be taken as $v(y) = |y|$, and in (1.5) we have $F_4 = 0$, $F_3 = b\dot x(t-h)$. Then

$$V_1 = |x(t)|, \quad \dot V_1 \leqslant -a|x(t)| + |b\dot x(t-h)| \tag{1.8}$$

Therefore

$$V_2 = c\int_{t-h}^{t} |\dot x(s)|\,ds, \quad c = \frac{|b|}{1-|b|} \tag{1.9}$$

Consequently, taking (1.8) and (1.9) into account and putting $V = V_1 + V_2$, we have

$$\dot V \leqslant -a|x(t)| + c|\dot x(t)| + |b\dot x(t-h)| - c|\dot x(t-h)| = -a|x(t)| + c|\dot x(t)| - bc|\dot x(t-h)|$$

Replacing $\dot x(t)$ by the right-hand side of Eq. (1.6), we obtain

$$\dot V \leqslant (-a + |a|c)|x(t)|$$

Hence, in view of (1.8) and (1.9), it is obvious that when $a > 0$, $|b| < 1/2$ the functional $V = V_1 + V_2$ is positive definite and its derivative is negative definite. Thus, when the functionals (1.8) and (1.9) are used, the asymptotic stability conditions (1.7) for system (1.6) remain unchanged.

We shall now consider some classes of systems (1.1), confining our attention, for simplicity, to a few characteristic representatives. Note that the aim of this study is not only to construct stability conditions but also to illustrate possible ways of implementing the individual steps of the procedure and the functionals V thus constructed.

2. LINEAR SYSTEMS

Using the procedure just described, let us establish stability conditions for the linear systems

$$\dot x(t) = A_0 x(t) + A_1 x(t-h_1) + A_2 \dot x(t-h_2), \quad t \geqslant 0, \quad x \in \mathbb{R}^n \tag{2.1}$$

where A_i are constant $n \times n$ matrices and $h_i \geqslant 0$ are constants. Let $\|\cdot\|$ denote the matrix norm induced by the vector norm $|\cdot|$ in $\mathbb{R}^n$, i.e. $\|A_i\|, = \max_x |A_i x|$, where the maximum is evaluated over all $x \in \mathbb{R}^n$ such that $|x| = 1$. Set $\alpha_i = \|A_i\|$. We will now consider possible versions of representations (1.4) and (1.5) and the resulting stability conditions and functionals $V = V_1 + V_2$, assuming that $\alpha_2 < 1$.

2.1. Define $F_1 = A_0 x$, $F_2 = A_1 x(t-h_1) + A_2 \dot x(t-h_2)$, $F_4 = 0$. Then the auxiliary equation is $\dot y(t) = A_0 y(t)$, the Lyapunov function v for which may be taken as $v = |y|$. This implies $V_1 = |x(t)|$. For any norm $|\cdot|$ in $\mathbb{R}^n$ we have [13]

$$d|x(t)|/dt = Q[x(t), \dot x(t)] \tag{2.2}$$

where the scalar function $Q[x, y]$ of two independent variables $x, y \in \mathbb{R}^n$ is defined by

$$Q[x,y] = \lim_{h\to 0^+} \frac{1}{h}[|x+hy| - |x|] \tag{2.3}$$

We recall, moreover, that for any square matrix A there exists

$$\gamma(A) = \sup_x |x|^{-1} Q[x, Ax], \quad |x| \neq 0, \quad x \in \mathbb{R}^n \tag{2.4}$$

where $\gamma(A)$ is the logarithmic norm of A [14]. It follows from (2.1)–(2.3) that

$$\dot{V}_1 \leq \lim_{h \to 0^+} \frac{1}{h}[\,|x + hy| - |x|\,] \leq \lim_{h \to 0^+} \frac{1}{h}[\,|x + hA_0 y| - |x|\,] + |F_2(t, x_t, \dot{x}_t)| =$$

$$= Q[x, A_0 x] + |F_2(t, x_t, \dot{x}_t)|$$

Hence, by (2.4), we obtain the inequality

$$\dot{V}_1 \leq \gamma(A_0)|x| + \alpha_1|x(t-h_1)| + \alpha_2|\dot{x}(t-h_2)| \tag{2.5}$$

To ensure that $\dot{V}$ is negative definite, we define

$$V_2 = (1-\alpha_2)^{-1}\left[\alpha_1 \int_{t-h_1}^{t} |x(s)|\,ds + \alpha_2 \int_{t-h_2}^{t} |\dot{x}(s)|\,ds\right] \tag{2.6}$$

In view of (2.5) and (2.6)

$$\dot{V} \leq [\gamma(A_0) + \alpha_1(1-\alpha_2)^{-1}]\,|x(t)| + \alpha_2(1-\alpha_2)^{-1}|\dot{x}(t)| -$$

$$-\alpha_1\alpha_2(1-\alpha_2)^{-1}|x(t-h_1)| - \alpha_2^2(1-\alpha_2)^{-1}|\dot{x}(t-h_2)|$$

By this inequality and the estimate implied by (2.1), we have

$$|\dot{x}(t)| \leq \alpha_0|x(t)| + \alpha_1|x(t-h_1)| + \alpha_2|\dot{x}(t-h_2)|$$

We finally conclude that $\dot{V} \leq \alpha_3|x(t)|$, where

$$\alpha_3 = \gamma(A_0) + (1-\alpha_2)^{-1}(\alpha_1 + \alpha_0\alpha_2)$$

We have thus shown that system (2.1) is asymptotically stable if

$$\alpha_3 < 0, \quad \alpha_2 < 1 \tag{2.7}$$

2.2. Let

$$F_1 = (A_0 + A_1)\,x, \quad F_4 = 0, \quad F_2 = F_3$$

$$F_2 = -A_1 \int_{t-h_1}^{t} \dot{x}(s)ds + A_2\dot{x}(t-h_2) \tag{2.8}$$

Then the auxiliary equation is $\dot{y} = (A_0 + A_1)\,y(t)$ and the Lyapunov function $\upsilon = |y|$, i.e. in accordance with the procedure and formulae (2.8) we must define $V_1 = |x(t)|$. Then, proceeding as for (2.5), we obtain

$$\dot{V}_1 \leq \gamma(A_0 + A_1)|x(t)| + \alpha_1 \int_{t-h_1}^{t} |\dot{x}(s)|\,ds + \alpha_2|\dot{x}(t-h_2)| \tag{2.9}$$

Hence, as in the case of (2.6), we obtain an expression for V_2

$$V_2 = (1 - h_1\alpha_1 - \alpha_2)^{-1}\left[\alpha_1 \int_{t-h_1}^{t} (s - t + h_1)\,|\dot{x}(s)|\,ds + \alpha_2 \int_{t-h_2}^{t} |\dot{x}(s)|\,ds\right]$$

Consequently

$$\dot{V}_2 = (1 - h_1\alpha_1 - \alpha_2)^{-1}\left[(h_1\alpha_1 + \alpha_2)|\dot{x}(t)| - \alpha_1 \int_{t-h_1}^{t} |\dot{x}(s)|\,ds - \alpha_2|\dot{x}(t-h_2)|\right]$$

Replacing $|\dot{x}(t)|$ here by the right-hand side of the estimate

$$|\dot{x}(t)| \leqslant \|A_0 + A_1\| |\dot{x}(t)| + \alpha_1 \int_{t-h_1}^{t} |\dot{x}(s)| ds + \alpha_2 |\dot{x}(t-h_2)|$$

and taking (2.9) into consideration, we conclude that $\dot{V} \leqslant \alpha_4 |x(t)|$, where

$$\alpha_4 = \gamma(A_0 + A_1) + \|A_0 + A_1\| (1 - h_1\alpha_1 - \alpha_2)^{-1} (h_1\alpha_1 + \alpha_2)$$

Thus, system (2.1) is asymptotically stable if

$$1 > h_1\alpha_1 + \alpha_2, \quad \alpha_4 < 0 \tag{2.10}$$

Conditions (2.7) and (2.10) are not identical.

Thus, for example, conditions (2.7) mean that A_0 is a Hurwitz matrix; conditions (2.10), however, may be satisfied even when all the eigenvalues of A_0 have positive real parts. It should also be noted that, under conditions (2.7), system (2.1) is stable for any delays h_1, $h_2 \geqslant 0$, while inequalities (2.10) impose certain restrictions on h_1.

Using other representations of the right-hand side of (2.1), one can deduce other stability conditions. To illustrate, here is one of them.

2.3. Put

$$F_1 = (A_0 + A_1)x, \quad F_3 = 0, \quad F_2 = \frac{d}{dt} F_4$$
$$F_4 = -A_1 \int_{t-h_1}^{t} |x(s)| ds + A_2 x(t-h_2) \tag{2.11}$$

The auxiliary system is $\dot{y} = (A_0 + A_1)y(t)$, the Lyapunov function $v(y)$ for which may be taken as the quadratic form $v(y) = y'By$, where the prime denotes transposition and B is a matrix such that

$$(A_0 + A_1)'B + B(A_0 + A_1) = -C \tag{2.12}$$

Let us assume that $A_0 + A_1$ is a Hurwitz matrix and the constant matrix C is positive definite. Then Eq. (2.12) has a unique solution $B > 0$ for any given matrix $C > 0$. In accordance with the procedure, we must put $V_1 = v(z)$, $z = x(t) - F_4$. Choose $C = I$, where I is the identity matrix; we will work with the Euclidean norm $|x| = (x'x)^{1/2}$. Then, taking (2.11) and (2.12) into account, we have

$$\dot{V}_1 = x'(t)(A_1 + A_0)'Bz(t) + z'(t)B(A_0 + A_1)x(t) = -|x(t)|^2 - 2x'(t)(A_0 + A_1)'BF_4$$

Thus V_2 should be chosen in the form

$$V_2 = \|(A_0 + A_1)B\| \left[\alpha_1 \int_{t-h_1}^{t} (s - t + h_1)|\dot{x}(s)|^2 ds + \alpha_2 \int_{t-h_2}^{t} |\dot{x}(s)|^2 ds\right]$$

The sum $V = V_1 + V_2$ then satisfies an estimate $\dot{V} \leqslant \alpha_5 |x(t)|^2$, where $\alpha_5 = -[1 - 2\|(A_0 + A_1)B\| (h_1\alpha_1 + \alpha_2)]$.

Since the functional V is only sign-definite, we must still require the trivial solution of the functional equation $x(t) - F_4(t, x_t) = 0$ to be asymptotically stable. A sufficient condition to that end is $h_1\alpha_1 + \alpha_2 < 1$. Consequently, system (2.1) will be asymptotically stable if

$$\operatorname{Re}\lambda(A_0 + A_1) < 0, \quad h_1\alpha_1 + \alpha_2 < 1, \quad \alpha_5 < 0 \tag{2.13}$$

where $\lambda(A_0 + A_1)$ are the eigenvalues of the matrix $A_0 + A_1$ and $\operatorname{Re}\lambda$ are their real parts.

We have thus proved the following theorem.

Theorem 1. System (2.1) is asymptotically stable if one of conditions (2.7), (2.10) or (2.13) is satisfied.

3. THE STABILITY OF A PREDATOR–PREY SYSTEM

The interaction of two populations of densities $N_1(t)$ and $N_2(t)$ has been simulated using the equations

$$\dot N_i(t) = N_i(t)\left[r_i - \sum_{j=1}^{2}\left(b_{ij}N_j(t-h_{ij}) - \gamma_{ij}\dot N_j(t-\tau_{ij})\right)\right], \quad t \geqslant 0 \tag{3.1}$$

with given constants γ_{ij} and non-negative constants $r_i, b_{ij}, h_{ij}, \tau_{ij}, i, j = 1, 2$ [7]. The initial conditions for system (3.1) are

$$N_i(\theta) = \varphi_i(\theta), \quad \dot N_i(\theta) = \dot\varphi_i(\theta), \quad \theta \leqslant 0$$

Let us assume that system (3.1) has a non-zero equilibrium position $N_i^0 > 0$, defined by the relations

$$\sum_{j=1}^{2} b_{ij}N_j^0 = r_i, \quad i = 1, 2$$

We shall consider the question of the stability of a solution N_i^0. By [2, 9], N_i^0 is asymptotically stable if the system, linearized with respect to the equilibrium position of N_i^0, is asymptotically stable. Setting $N_i = m_i + N^0$, we see that the linearized system is

$$\dot m_i(t) + \sum_{j=1}^{2}[P_{ij}\dot m_i(t-\tau_{ij}) + a_{ij}m_i(t-h_{ij})] = 0 \tag{3.2}$$

$$P_{ij} = b_{ij}N_j^0, \quad a_{ij} = \gamma_{ij}N_j^0$$

Now let us use our procedure to establish stability conditions for system (3.2). Let $\delta(s)$ denote a function equal to zero for $s \leqslant 0$ and to unity for $s > 0$. We write the characteristic equation corresponding to (3.2)

$$z^2 = -\int_0^\infty e^{-zs}[z^2 dK_3(s) + z\,dK_2(s) + dK_1(s)] \tag{3.3}$$

where z is a complex variable, the integrals are understood in the Lebesgue–Stieltjes sense and we have defined

$$K_1(s) = a_{11}a_{22}\delta(s-h_{11}-h_{22}) - a_{12}a_{21}\delta(s-h_{12}-h_{21}), \quad s \geqslant 0 \tag{3.4}$$

$$K_2(s) = a_{22}\delta(s-h_{22}) - a_{22}p_{11}\delta(s-\tau_{11}-h_{22}) + a_{11}\delta(s-h_{11}) + a_{11}p_{22}\delta(s-h_{11}-\tau_{22}) -$$
$$-p_{12}a_{21}\delta(s-\tau_{12}-h_{21}) - a_{12}p_{21}\delta(s-h_{12}-\tau_{21})$$

$$K_3(s) = p_{22}\delta(s-\tau_{22}) + p_{11}\delta(s-\tau_{11}) + p_{11}p_{22}\delta(s-\tau_{11}-\tau_{22}) + p_{12}p_{21}\delta(s-\tau_{12}-\tau_{21})$$

Equation (3.3) is also the characteristic equation for the second-order system

$$\dot x_1(t) = x_2(t), \quad \dot x_2(t) = -\int_0^\infty [\dot x_2(t-s)\,dK_3(s) + x_2(t-s)\,dK_2(s) + x_1(t-s)\,dK_1(s)] \tag{3.5}$$

Let us assume that

$$\int_0^\infty |dK_3(s)| < 1 \tag{3.6}$$

If this inequality holds, the necessary and sufficient conditions for the asymptotic stability of systems (3.2) and (3.5) are the same: Eq. (3.3) has no roots in the right-hand half-plane for $\operatorname{Re} z \geqslant 0$. It will therefore suffice to establish stability conditions for system (3.5), and to that end we will use the procedure described previously.

Note that because of (3.4) all the kernels $K_i(s)$ have bounded moments. Put

$$\alpha_{ij}=\int_0^\infty s^i|dK_j(s)|,\quad \beta_{ij}=\int_0^\infty s^i dK_j(s)$$

The integrals in (3.5) may be expressed as follows:

$$\int_0^\infty x_2(t-s)\,dK_2(s)=-\frac{d}{dt}\int_0^\infty dK_2(s)\int_{t-s}^t x_2(\tau)\,d\tau+\beta_{02}x_2(t)=\frac{d}{dt}\int_0^\infty x_1(t-s)\,dK_2(s) \tag{3.7}$$

$$\int_0^\infty x_1(t-s)\,dK_1(s)=\frac{d}{dt}\int_0^\infty dK_1(s)\int_{t-s}^t(\tau-t+s)\,x_2(\tau)\,d\tau-\beta_{11}x_2(t)+\beta_{01}x_1(t)=$$

$$=-\frac{d}{dt}\int_0^\infty dK_1(s)\int_{t-s}^t x_1(\tau)\,d\tau+\beta_{01}x_1(t)=-\int_0^\infty dK_1(s)\int_{t-s}^t x_2(\tau)\,d\tau+\beta_{01}x_1(t). \tag{3.8}$$

Using different representations (3.7) and (3.8), we obtain different transformed systems.

We will consider one of them, since the others can be investigated similarly. Using (3.7) and (3.8), we write system (3.5) in the form

$$\dot{x}_1(t)=x_2(t),\quad \dot{z}(t)=-a\,x_2(t)-b\,x_1(t) \tag{3.9}$$

$$a=\beta_{02}-\beta_{11},\quad b=\beta_{01}$$

$$z(t)=x_2(t)+\int_0^\infty\left[x_1(t-s)\,dK_3(s)-dK_2(s)\int_{t-s}^t x_2(\tau)\,d\tau+dK_1(s)\int_{t-s}^t(\tau-t+s)\,x_2(\tau)\,d\tau\right]$$

Based on (3.9), the auxiliary system is

$$\dot{y}_1(t)=y_2(t),\quad \dot{y}_2(t)=-a\,y_2(t)-b\,y_1(t) \tag{3.10}$$

The Lyapunov function for system (3.10) is given by

$$\upsilon=(2ab)^{-1}[(a^2+b^2+b)\,y_1^2+2ay_1y_2+(b+1)\,y_2^2]$$

Thus, in accordance with the procedure, we have $V=V_1+V_2$, where

$$V_1=(2ab)^{-1}[(a^2+b^2+b)\,x_1^2(t)+2ax_1(t)\,z(t)+(b+1)\,z^2(t)] \tag{3.11}$$

To construct the component V_2, we calculate $\dot{V}_1$ along trajectories of system (3.9)

$$\dot{V}_1=-x_1^2(t)-x_2^2(t)-(z(t)-x_2(t))\,(x_2(t)+a^{-1}(b+1)\,x_1(t))$$

Note further that by (3.9)

$$2\,|(z(t)-x_2(t))\,x_2(t)|\ \leqslant\ q\,x_2^2(t)+J,\quad q=\alpha_{03}+\alpha_{12}+\alpha_{21}/2$$

$$J=\int_0^\infty\left[x_2^2(t-s)\,|dK_3(s)|+|dK_2(s)|\int_{t-s}^t x_2^2(\tau)\,d\tau+|dK_1(s)|\int_{t-s}^t(\tau-t+s)\,x_2^2(\tau)\,d\tau\right]$$

$$2\,|(z(t)-x_2(t))\,x_1(t)|\ \leqslant\ q\,x_1^2(t)+J$$

We must therefore define

$$V_2=\frac{b+1+a}{2a}\int_0^\infty\left[|dK_3(s)|\int_{t-s}^t x_2^2(\tau)\,d\tau+|dK_2(s)|\int_{t-s}^t(\tau-t+s)\,x_2^2(\tau)\,d\tau+\right.$$

$$\left.+|dK_1(s)|\int_{t-s}^t\frac{(\tau-t+s)}{2}\,x_2^2(\tau)\,d\tau\right]$$

The sum $V = V_1 + V_2$ then satisfies the estimate

$$\dot{V} \leqslant -x_1^2(t)\left(1-\frac{b+1}{2a}q\right)-x_2^2(t)\left(1-\frac{b+1+2a}{2a}q\right) \tag{3.12}$$

The functional V we have constructed is only sign-definite. We must, therefore, still require the trivial solution of the functional equation $z = 0$ to be asymptotically stable. A sufficient condition for this to be true is $q < 1$. Hence, since (3.12) is negative definite, it follows that the equilibrium position N_i^0 of system (3.1) is asymptotically stable if

$$q=\alpha_{03}+\alpha_{12}+\alpha_{21}/2<1,\quad \beta_{01}>0,\quad 2(\beta_{02}-\beta_{11})(1-q)>(1+\beta_{01})q \tag{3.13}$$

Note that at $q = 0$ conditions (3.13) are identical with the necessary and sufficient conditions for system (3.10) to be stable. Combining representations (3.7) and (3.8) differently, or choosing other Lyapunov functions for the auxiliary equation, one can derive other stability conditions for system (3.1). For example, the transformed system has the form (3.9) if

$$a=-\beta_{11},\quad b=\beta_{01},\quad z(t)=x_2(t)+ \tag{3.14}$$

$$+\int_0^\infty [x_2(t-s)\,dK_3(s)-dK_2(s)\,x_1(t-s)+dK_1(s)\int_{t-s}^{t}(\tau-t+s)\,x_2(\tau)\,d\tau]$$

With the a, b and z values of (3.14), the functional V_1 is defined by (3.11) and the functional V_2 is

$$V_2=\frac{b+1+a}{2a}\int_0^\infty\Bigg[|dK_3(s)|\int_{t-s}^{t}x_2^2(\tau)\,d\tau+|dK_2(s)|\int_{t-s}^{t}x_1^2(\tau)\,d\tau+$$

$$+|dK_1(s)|\int_{t-s}^{t}\frac{(\tau-t+s)^2}{2}x_2^2(\tau)\,d\tau\Bigg]$$

Evaluating $\dot{V}$ along trajectories of system (3.5), we get

$$\dot{V} \leqslant -x_1^2(t)-x_2^2(t)+\frac{1}{2}x_1^2(t)\left(\alpha_{02}+\frac{b+1}{a}\left(2\alpha_{02}+\frac{\alpha_{21}}{2}+\alpha_{03}\right)\right)+$$

$$+\frac{1}{2}x_2^2(t)\left(\alpha_{02}+\alpha_{21}+\frac{b+1}{2a}\alpha_{21}+\frac{b+1+2a}{2a}\alpha_{03}\right)$$

Consequently, the conditions for the equilibrium position of system (3.1) to be stable are

$$-\beta_{11}>0,\quad \beta_{01}>0$$

$$-2\beta_{11}>\max\Bigg[-\beta_{11}\alpha_{02}+(1+\beta_{01})\left(2\alpha_{02}+\frac{\alpha_{21}}{2}+\alpha_{03}\right),$$

$$-\beta_{11}(\alpha_{02}+\alpha_{21})+(1+\beta_{01})\frac{\alpha_{21}}{2}+\frac{\beta_{01}+1-2\beta_{11}}{2}\alpha_{03}\Bigg]$$

4. NON-LINEAR SYSTEMS

We will now describe the application of the procedure to non-linear systems, establishing stability conditions for them, confining ourselves for simplicity to Eqs (1.1) of the form

$$\dot{x}(t)=A_1(t,x(t-h_1))+A_2(t,\dot{x}(t-h_2)),\quad t \geqslant t_0,\quad x(t)\in\mathbb{R}^n \tag{4.1}$$

with initial conditions (1.2).

Here $h_i \geqslant 0$ are constants and A_1: $[t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $A_i(t, 0) = 0$ $(i = 1, 2)$ are continuous functions. Let D denote the set of absolutely continuous functions ψ: $(-\infty, 0] \to \mathbb{R}^n$ such that $|\psi| < R$ for some $R > 0$. Let us assume that the function $A_1(t, x)$ is continuously differentiable with respect to x, and for some constants $\alpha_i > 0$ and any $\varphi, \psi \in D$ we have

$$\begin{aligned} \|A_2(t, \dot{\psi}(-h_2))\| &\leqslant \alpha_2 \|\dot{\psi}(-h_2)\|, \quad \alpha_2 < 1 \\ \|A_1(t, \varphi(-h_1)) - A_1(t, \psi(-h_1))\| &\leqslant \alpha_1 \|\varphi(-h_1) - \psi(-h_1)\| \end{aligned} \tag{4.2}$$

Let $f(t, x)$: $[t_0, \infty)$ $\mathbb{R}^n \to \mathbb{R}^n$ denote the Jacobian, $f(t, x) = \partial A_1(t, x)/\partial x$, and $\gamma(f)$ the logarithmic norm of the matrix f. Let $\gamma_1 = \sup_\alpha \gamma(f(t, x))$, $t \geqslant t_0$, x: $|x| \leqslant r$.

Theorem 2. Suppose that inequalities (4.2) hold and moreover

$$\gamma_2 < 0, \quad \alpha_2 + \alpha_1 h_1 < 1, \quad \gamma_2 = \gamma_1 + \alpha_1(\alpha_1 h_1 + \alpha_2)(1 - \alpha_2 - \alpha_1 h_1)^{-1} \tag{4.3}$$

Then the trivial solution of system (4.1) is asymptotically stable.

Proof. Let $F_1 = A_1(t, x(t))$. This means that the auxiliary system has the form $y(t) = A_1(t, y(t))$. Let $v = |y|$. Then in formula (1.4) we must put $F_3 = A_1(t, x(t - h_1)) - A_1(t, x(t)) + A_2(t, \dot{x}(t - h_2))$, $F_4 = 0$. Consequently, $V_1 = |x(t)|$. Therefore, by (2.2) and (2.3)

$$\dot{V}_1 \leqslant \lim_{h \to 0^+} \frac{1}{h} [\,|x(t) + hA_1(t, x(t))| - |x(t)|\,] + |F_3| \tag{4.4}$$

Note that

$$A_1(t, x) = \left[\int_0^1 f(t, sx)\, ds\right] x$$

Hence, from (4.4), we obtain the inequality

$$\dot{V}_1 \leqslant \lim_{h \to 0^+} \frac{1}{h}\left[\left|x(t) + h\left(\int_0^1 f(t, sx)\, ds\right) x(t)\right| - |x(t)|\right] + |F_3| \tag{4.5}$$

In view of (2.3), (2.4) and (4.5), we have

$$\begin{aligned} \dot{V}_1 &\leqslant Q\left[x(t), \left(\int_0^1 f(t, sx(t))\, ds\right) x(t)\right] + |F_3| \leqslant \\ &\leqslant \gamma\left(\int_0^1 f(t, sx(t))\, ds\right)|x(t)| + |F_3| \leqslant \int_0^1 \gamma(f(t, sx(t)))\, ds\, |x(t)| + |F_3| \end{aligned} \tag{4.6}$$

Now consider the functional $V = V_1(\psi) + V_2(\psi)$ in D, where

$$\begin{aligned} V_2(\psi) = (1 - \alpha_2 - \alpha_1 h_1)^{-1} \Bigg[&\alpha_1 \int_{-h_1}^0 (\tau + h_1) |\dot{\psi}(\tau)|\, d\tau + \\ &+ \alpha_2 \int_{-h_2}^0 |\dot{\psi}(\tau)|\, d\tau + (\alpha_1 h_1 + \alpha_2)\alpha_1 \int_{-h_1}^0 |\psi(\tau)|\, d\tau \Bigg] \end{aligned} \tag{4.7}$$

On the basis of (4.2) and (4.6)

$$V_1(\psi) \leqslant \gamma_1|\psi(0)|+\alpha_2|\dot\psi(-h_2)|+\alpha_1|\psi(-h_1)-\psi(0)| \leqslant \\ \leqslant \gamma_1|\psi(0)|+\alpha_2|\dot\psi(-h_2)|+\alpha_1\int_{-h_1}^{0}|\dot\psi(\tau)|d\tau \tag{4.8}$$

We now calculate $\dot V_2(\psi)$

$$\dot V_2(\psi)=(1-\alpha_2-\alpha_1 h_1)^{-1}\Big[(\alpha_1 h_1+\alpha_2)|\dot\psi(0)|-\alpha_1\int_{-h_1}^{0}|\dot\psi(\tau)|d\tau-\alpha_2|\dot\psi(-h_2)|+ \\ +(\alpha_1 h_1+\alpha_2)\alpha_1(|\psi(0)|-|\psi(-h_1)|)\Big] \tag{4.9}$$

Note that, by (4.1) and (4.2)

$$|\dot\psi(0)| \leqslant \alpha_1|\psi(-h_1)|+\alpha_2|\dot\psi(-h_2)| \tag{4.10}$$

Replacing $|\dot\psi(0)|$ in (4.9) by the right-hand side of (4.10) and taking note of (4.8), we obtain $\dot V(\psi) \leqslant \gamma_2|\psi(0)|$. Thus, V is positive definite in D, admits of an infinitely small upper limit, and its total derivative is negative definite when conditions (4.2) and (4.3) hold. Consequently [3, 9], the trivial solution of system (4.1) is asymptotically stable.

Remark 1. In applications of the procedure to equations of neutral type it is sometimes useful to iterate the derivative on the right once or more times. For example, define $V_1 = |x(t)|$ for $t \geqslant t_0 + h_1$. Then, as in (4.8) and (4.10), integrating once, we conclude that

$$\dot V_1(\psi) \leqslant \gamma_1|x(t)|+\alpha_2|\dot x(t-h_2)|+\alpha_1\int_{t-h_1}^{t}|\dot x(\tau)|d\tau \leqslant \\ \leqslant \gamma_1|x(t)|+\alpha_2|\dot x(t-h_2)|+\alpha_1^2\int_{t-2h_1}^{t-h_1}|x(\tau)|d\tau+\alpha_1\alpha_2\int_{t-h_1-h_2}^{t-h_2}|\dot x(\tau)|d\tau$$

Thus

$$V_2=\alpha_1^2\left[\int_{t-2h_1}^{t-h_1}ds\int_{s}^{t-h_1}|x(s)|ds+h_1\int_{t-h_1}^{t}|x(\tau)|d\tau\right]+ \\ +\alpha_2(1-\alpha_2(1+\alpha_1 h_1))^{-1}\left[\int_{t-h_2}^{t}|\dot x(\tau)|d\tau+\int_{t-h_1}^{t}|x(\tau)|d\tau\alpha_1\right]+\alpha_1\alpha_2\int_{t-h_1-h_2}^{t-h_2}ds\int_{s}^{t-h_2}|\dot x(s)|ds$$

Setting $V = V_1 + V_2$, we obtain $\dot V(x) \leqslant \gamma_3|x(t)|$, where

$$\gamma_3=\gamma_1+\alpha_1^2 h_1+\alpha_2(1-\alpha_2(1+\alpha_1 h_1))^{-1}\alpha_1$$

We have thus proved the following theorem.

Theorem 3. The trivial solution of system (4.1) is asymptotically stable if inequalities (4.2) hold and $\gamma_3 < 0$, $(\alpha_1 h_1 + 1)\alpha_2 < 1$.

Remark 2. For some systems of type (1.1) one has $V = V_1$. For example, suppose that the transformations (1.4) and (1.5) reduce Eq. (1.1) to the form $\dot z = F_5(t, z(t))$, where $z(t) = x(t) - F_4(t, x_t)$, $F_4(t, 0) = 0$, $F_5(t, 0) = 0$. Then the auxiliary equation has the form $\dot y = F_5(t, y(t))$, with Lyapunov function $v(t, y)$. Consequently, $V = V_1 = v(t, z(t))$. Therefore, if the trivial solution of the equation $\dot y = F_5(t, y(t))$, is uniformly asymptotically stable and the trivial solution of the equation $x(t) = F_4(t, x_t)$ is asymptotically stable, then the trivial solution of system (1.1) is also asymptotically stable. This was pointed out in [3].

Example 2. Consider a shunted power transmission line, as described by the equation [2]

$$x(t)=-g(x(t))+b\dot x(t-h),\ t \geqslant 0,\ x\in\mathbb{R},\ |b|<1,\ g(0)=0 \tag{4.11}$$

where $g(x)$: $\mathbb{R} \to \mathbb{R}$ is a continuous function, $b, h \geqslant 0$ are given constants, and

$$x\,g(x) > 0, \quad x \neq 0 \tag{4.12}$$

The stability of the trivial solution of (4.11) was studied in [2], where it was established that sufficient conditions for stability are provided by inequalities (4.12) and $|b| < 1/2$. If $g(x)$ is a linear function, i.e. $g(x) = ax$, these conditions reduce to $a > 0$, $|b| < 1/2$, i.e. they are not the same as (1.7), which were applied to the linear equation (1.6) in Example 1. Using the procedure proposed here, we will establish stability conditions for the trivial solution of system (4.11) that reduce to (1.7) in the linear case. The auxiliary ordinary equation for (4.11) has the form $\dot{y}(t) = -g(y(t))$, a Lyapunov function for which may be taken as

$$v(y) = \int_0^y g(s)\,ds \tag{4.13}$$

By (4.12), $v(y)$ is positive definite and has an infinitesimal upper limit. On the basis of the procedure, $V_1 = v(x(t))$. Hence, via (4.11), it follows that $V = V_1 + V_2$ where

$$V_2 = \frac{1}{2}\int_{t-h}^{t} \dot{x}^2(\tau)\,d\tau \tag{4.14}$$

One then finds that the derivative $\dot{V}$ is given by

$$\dot{V} = g(x(t))[-g(x(t)) + b\dot{x}(t-h)] + \frac{1}{2}(\dot{x}^2(t) - \dot{x}^2(t-h))$$

Replacing $\dot{x}(t)$ here by the right-hand side of Eq. (4.11), we obtain

$$2\dot{V} = -g^2(x(t)) - (1-b^2)\dot{x}^2(t-h)$$

Thus, the asymptotic stability conditions are the inequalities (4.12) and $|b| < 1$, which become (1.6) in the linear case $g(x) = ax$.

Equation (4.11) with a non-linearity $g(x)$ satisfying (4.12) is an equation of gradient type, for which application of the above procedure also yields stability conditions for $n > 1$.

For example, consider the system

$$\dot{x}(t) = -\nabla G(x(t)) + B\dot{x}(t-h), \quad t \geqslant t_0, \quad x(t) \in \mathbb{R}^n, \quad G(0) = 0 \tag{4.15}$$

where $G(x)$: $\mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, B is a given constant matrix, $h \geqslant 0$, ∇G denotes the gradient of G and

$$G(x) > 0, \quad x \neq 0 \tag{4.16}$$

As in (4.13) and (4.14), putting

$$V(x) = G(x) + \frac{1}{2}\int_{t-h}^{t} \dot{x}'(s)\,\dot{x}(s)\,ds$$

we conclude that the trivial solution of system (4.15) is asymptotically stable if inequality (4.16) holds and $\| B \| < 1$.

Remark 3. The procedure is also applicable to certain systems in an unsteady state. For example, consider the scalar equation

$$\dot{x}(t) = -a(t)\,x(t) + b\,\dot{x}(t-h), \quad t \geqslant t_0 \tag{4.17}$$

where $a(t)$ is a non-negative continuous function, $b, h \geqslant 0$ are given constants, and $\| b \| < 1$. Then the auxiliary equation is $\dot{y}(t) = -a(t)\,y(t)$, for which a Lyapunov function v may be taken as $v = |y|$. Thus, $V_1 = |x(t)|$. Therefore

$$V_2 = \frac{|b|}{1-|b|}\int_{t-h}^{t} |\dot{x}(s)|\,ds$$

For $V = V_1 + V_2$ we have $\dot{V} \leqslant -a(t)|x(t)| + |b|(1-|b|)^{-1}(|\dot{x}(t)| - |\dot{x}(t-h)|)$. Replacing $|x(t)|$ here, in accordance with (4.17), by $|a(t)\,x(t)| + |b\dot{x}(t-h)|$, we conclude, using [9], that system (4.17) is asymptotically stable if $a(t) \geqslant c > 0$, $|b| < 1/2$.

Using a different function

$$V=[x(t)+bx(t-h)]^2+|b|\int_{t-h}^{t} a(s+h)\,x^2(s)\,ds$$

we arrive at the following asymptotic stability conditions

$$2a(t)-|b|(a(t)+a(t+h)) \geqslant c>0, \quad |b|<1$$

These conditions reduce to the stability conditions (1.7) for system (1.6) with constant coefficients.

REFERENCES

1. BELLMAN R. and COOKE K., *Differential-difference Equations*. Mir, Moscow, 1967.
2. HALE J. K. and LUNEL S. M. V., *Introduction to Functional Differential Equations*. Springer, Berlin, 1993.
3. KOLMANOVSKII V. B. and NOSOV V. P., Hereditary systems of neutral type. *Avtomatika i Telemekhanika* 1, 5–35, 1984.
4. BELOTSERKOVSKII S. M., KOCHETKOV Yu. A., KRASOVSKII A. A. and NOVITSKII V. V., *Introduction to Aero-autoelasticity*. Nauka, Moscow, 1980.
5. BOGOMOLOV V. L., Automatic control of hydroelectric station output by load shedding. *Avtomatika i Telemekhanika* 4/5, 103–129, 1941.
6. BRAYTON R. K., Nonlinear oscillations in a distributed network. *Quart. Appl. Math.* **24**, 4, 289–301, 1967.
7. KUANG Y., *Delay Differential Equations with Applications in Population Dynamics*. Academic Press, San Diego, CA, 1993.
8. GOPALSAMY K., *Equations of Mathematical Ecology*. Kluwer, Dordrecht, 1992.
9. KOLMANOVSKII V. B. and MYSHKIS A. D., *Applied Theory of Functional Differential Equations*. Kluwer, Dordrecht, 1992.
10. KRASOVSKII N. N., *Some Problems of the Theory of Stability of Motion*. Fizmatgiz, Moscow, 1959.
11. KIM A. V., *Lyapunov's Direct Method in the Theory of the Stability of Hereditary Systems*. Izd. Ural. Univ., Ekaterinburg, 1992.
12. KOLMANOVSKII V. B., The stability of certain systems with arbitrary aftereffect. *Dokl. Ross. Akad. Nauk* **331**, 4, 421–424, 1993.
13. BYLOV B. F., VINOGRAD R. E., GROBMAN D. M. and NEMYTSKII V. V., *Theory of Lyapunov Exponents and its Applications to Stability Problems*. Nauka, Moscow, 1966.
14. LOZINSKII S. M., Evaluation of the error of the numerical integration of ordinary differential equations. *Izv. Vuzov, Matematika* 5, 52–90, 1958.

Translated by D.L.